Periodica Mathematica Hungarica Vol. 62 (1), 2011, pp. 61–73
DOI: 10.1007/s10998-011-5061-8

NECESSARY AND SUFFICIENT CONDITIONS FOR THE STRONG LAW OF LARGE NUMBERS

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(*Received August 2, 2010; Accepted September 28, 2010*)

Dedicated to Professors Endre Csáki and Pál Révész
on the occasion of their 75th birthdays

Abstract

The well-known characterization indicated in the title involves the moving maximal dyadic averages of the sequence $(X_k : k = 1, 2, \ldots)$ of random variables in Probability Theory. In the present paper, we offer another characterization of the SLLN which does not require to form any maximum. Instead, it involves only a specially selected sequence of moving averages. The results are also extended for random fields $(X_{k\ell} : k, \ell = 1, 2, \ldots)$.

0. Background in probability theory

In the framework of Kolmogorov's axiomatic treatment of probability, one of the fundamental questions is the relationship between probability and relative frequency. The results of this investigation are called the *Laws of Large Numbers.* Similarly, the relationship between the expectation of a random variable and sample mean can also be studied by using the laws of large numbers. For example, the celebrated theorem of Kolmogorov on the *Strong Law of Large Numbers* reads as follows: Let $(X_k : k = 1, 2, \ldots)$ be a sequence of independent, and identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then the sequence of the *arithmetic averages*

$$\sigma_M := \frac{1}{M} \sum_{k=1}^{M} X_k, \quad M = 1, 2, \ldots,$$

Mathematics subject classification numbers: 60F15; 40B05, 40G99.
Key words and phrases: sequences of random variables, SLLN, moving averages, Toeplitz's theorem, random fields, moving rectangular averages, bounded convergence in Pringsheim's sense, Robison's theorem in Summability Theory.

0031-5303/2011/$20.00
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Springer, Dordrecht
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converges almost surely to some constant c if and only if the expectation EX_1 exists, in which case $c = EX_1$; in symbols:

$$P\Big[\lim_{M\to\infty} \frac{1}{M}\sum_{k=1}^{M} X_k(\omega) = EX_1\Big] = 1.$$

The well-known proof of this almost sure convergence result hinges on the estimate of the moving maximal dyadic averages of the sequence (X_k) in question. The interested reader is referred to the book [2] by Révész to get a comprehensive account of this classical subject.

Our primary aim in this paper is to discuss the SLLN from the viewpoint of Summability Theory. In this way, we will obtain a new necessary and sufficient condition for the validity of the SLLN, which does not require to form any maximum. Instead, it only involves a specially selected sequence of moving averages of the given sequence (X_k) of random variables.

1. Known characterizations

For the sake of brevity in writing, we will use the following notation:

$$I_1 := \{1\},\ I_2 := \{2\},\ I_m := \{2^{m-2}+1, 2^{m-2}+2, \dots, 2^{m-1}\} \quad \text{for} \quad m = 3, 4, \dots.$$

The next two theorems are folklore.

Theorem A. *Let $(X_k : k = 1, 2, \dots)$ be a sequence of random variables. Then*

$$P\Big[\lim_{m\to\infty} \frac{1}{2^m}\sum_{k=1}^{2^m} X_k = 0\Big] = 1$$

if and only if

$$P\Big[\lim_{m\to\infty} \frac{1}{|I_m|}\sum_{k\in I_m} X_k = 0\Big] = 1. \tag{1.1}$$

By $|I_m|$ we denote the number of integers in I_m, that is,

$$|I_1| = 1, \quad |I_m| = 2^{m-2} \quad \text{for} \quad m = 2, 3, \dots.$$

The ratio in (1.1) may be called the *moving dyadic average* of the sequence (X_k).

THEOREM B. *A sequence $(X_k : k = 1, 2, \ldots)$ of random variables obeys the SLLN, that is,*

$$P\Big[\lim_{m\to\infty} \frac{1}{M}\sum_{k=1}^{M} X_k = 0\Big] = 1 \tag{1.2}$$

if and only if

$$P\Big[\lim_{m\to\infty} \frac{1}{|I_m|} \max_{p\in I_m} \Big| \sum_{k=2^{m-2}+1}^{p} X_k\Big| = 0\Big] = 1, \tag{1.3}$$

where we agree to set $2^{m-2} := 0$ for $m = 1$.

Clearly, (1.2) is an easy consequence of (1.1) and (1.3). The ratio in (1.3) may be called the *moving maximal dyadic average* of the sequence (X_k).

We note that there is a misprint in the formulation of [2, Theorem 1.2.3a on p. 37], as the following example shows.

EXAMPLE 1. Let $X_1 := 1$, $X_2 := -1$,

$$X_k := 1 \quad \text{if} \quad 2^{m-1} < k < 2^m, \quad \text{and} \quad X_{2^m} := 1 - 2^{m-1} \quad \text{for} \quad m = 2, 3, \ldots.$$

Since

$$\frac{1}{2^m - 1}\sum_{k=1}^{2^m-1} X_k = \frac{2^{m-1}-1}{2^m - 1} \quad \text{and} \quad \frac{1}{2^m}\sum_{k=1}^{2^{m-1}} X_k = 0 \quad \text{for} \quad m = 2, 3, \ldots,$$

condition (1.1) is clearly satisfied, while (1.2) is not. The correct formulation of [2, Theorem 1.2.3 a] is given in Theorem A.

Before discussing analogous results for random fields, we consider a double sequence $(x_{k\ell} : k, \ell = 1, 2, \ldots)$ of complex numbers, whose rectangular averages are defined by

$$\sigma_{MN} := \frac{1}{MN}\sum_{k=1}^{M}\sum_{\ell=1}^{N} x_{k\ell}, \quad M, N = 1, 2, \ldots.$$

We recall that the double sequence $(x_{k\ell})$ is said to *converge in Pringsheim's sense* to 0 if for every $\varepsilon > 0$ there exists some natural number $k_0 = k_0(\varepsilon)$ such that

$$|x_{k\ell}| < \varepsilon \quad \text{whenever} \quad \min\{k, \ell\} > k_0.$$

It is well known that the convergence of a double sequence $(x_{k\ell})$ in Pringsheim's sense does not imply the boundedness of its terms (see Example 2 below). However, if a double sequence $(x_{k\ell})$ is bounded and converges to 0, then the double sequence (σ_{MN}) of its rectangular averages is also bounded and converges to 0. The restriction to bounded sequences is justified by the following

Example 2. Let

$$x_{k\ell} := \begin{cases} k, & \text{if } k = 1, 2, \ldots \text{ and } \ell = 1; \\ \ell, & \text{if } k = 1 \text{ and } \ell = 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the double sequence $(x_{k\ell})$ converges to 0 as $k, \ell \to \infty$. But the double sequence

$$\sigma_{MN} := \frac{1}{MN} \sum_{k=1}^{M} \sum_{\ell=1}^{N} x_{k\ell} = \frac{1}{MN} \Big(\sum_{k=1}^{M} k + \sum_{\ell=2}^{N} \ell \Big), \quad M, N = 1, 2, \ldots,$$

of its rectangular averages fails to have any limit. For example, check the particular cases when $M = N$ or $M = 2N$.

Therefore, in the sequel we will use the notion of *bounded convergence*, in symbols:

$$\mathrm{b-}\lim_{k,\ell\to\infty} x_{k\ell} = 0,$$

to indicate that the double sequence $(x_{k\ell})$ converges to 0 in Pringsheim's sense and

$$\sup_{k,\ell\ge 1} |x_{k\ell}| < \infty.$$

The following extensions of Theorems A and B to random fields $(X_{k\ell})$ are immediate consequences of [1, Theorems A* and B* in Section 5].

Theorem C. *Let $(X_{k\ell} : k, \ell = 1, 2, \ldots)$ be a random field. Then*

$$P\Big[\mathrm{b-}\lim_{m,n\to\infty} \frac{1}{2^m 2^n} \sum_{k=1}^{2^m} \sum_{\ell=1}^{2^n} X_{k\ell} = 0 \Big] = 1$$

if and only if

$$P\Big[\mathrm{b-}\lim_{m,n\to\infty} \frac{1}{|I_m| \cdot |I_n|} \sum_{k\in I_m} \sum_{\ell\in I_n} X_{k\ell} = 0 \Big] = 1. \tag{1.4}$$

The ratio in (1.4) may be called the *moving dyadic rectangular average* of $(X_{k\ell})$.

THEOREM D. *A random field* $(X_{k\ell} : k, \ell = 1, 2, \ldots)$ *obeys the SLLN, that is,*

$$P\Big[\operatorname*{b-lim}_{M,N\to\infty} \frac{1}{MN} \sum_{k=1}^{M} \sum_{\ell=1}^{M} X_{k\ell} = 0\Big] = 1 \tag{1.5}$$

if and only if

$$P\Big[\operatorname*{b-lim}_{m,n\to\infty} \frac{1}{|I_m| \cdot |I_n|} \max_{p\in I_m, q\in I_n} \Big| \sum_{k=2^{m-2}+1}^{p} \sum_{\ell=2^{n-2}+1}^{q} X_{k\ell} \Big| = 0\Big] = 1, \tag{1.6}$$

again with the agreement that $2^{m-2} := 0$ *for* $m = 1$.

The ratio in (1.6) may be called the *moving maximal dyadic rectangular average* of the random field $(X_{k\ell})$.

2. New results

In Theorems B and D in Section 1 above, the SLLN is characterized in terms of moving maximal averages. In this section, our goal is to characterize the SLLN only in terms of certain moving averages that do not require to form any maximum. To this effect, we will construct special sequences of moving averages in the case of single as well as double sequences.

Given a sequence $(X_k : k = 1, 2, \ldots)$ of random variables, we set

$$\sigma_m^{(1)} := \frac{1}{m} \sum_{k=m}^{2m-1} X_k \quad \text{for} \quad m = 1, 2, \ldots, \text{ and} \tag{2.1}$$

$$\sigma_1^{(2)} := \frac{1}{2} \sum_{k=1}^{2} X_k, \quad \sigma_m^{(2)} := \frac{1}{m} \sum_{k=m+1}^{2m} X_k \quad \text{for} \quad m = 2, 3, \ldots. \tag{2.2}$$

Observe that in the case of $\sigma_m^{(1)}$ the upper limit of the summation is an odd number, while in the case of $\sigma_m^{(2)}$ the upper limit of the summation is an even number. Clearly, these $\sigma_m^{(1)}$ and $\sigma_m^{(2)}$ are moving averages of the sequence (X_k).

We note that the sequences $(\sigma_m^{(1)})$ and $(\sigma_m^{(2)})$ may be merged into a single sequence, whose first fifteen terms are the following:

$$\frac{1}{1}\sum_{k=1}^{1} X_k, \quad \frac{1}{2}\sum_{k=1}^{2} X_k, \quad \frac{1}{2}\sum_{k=2}^{3} X_k, \quad \frac{1}{2}\sum_{k=3}^{4} X_k, \quad \frac{1}{3}\sum_{k=3}^{5} X_k,$$

$$\frac{1}{3}\sum_{k=4}^{6} X_k, \quad \frac{1}{4}\sum_{k=4}^{7} X_k, \quad \frac{1}{4}\sum_{k=5}^{8} X_k, \quad \frac{1}{5}\sum_{k=5}^{9} X_k, \quad \frac{1}{5}\sum_{k=6}^{10} X_k,$$

$$\frac{1}{6}\sum_{k=6}^{11} X_k, \quad \frac{1}{6}\sum_{k=7}^{12} X_k, \quad \frac{1}{7}\sum_{k=7}^{13} X_k, \quad \frac{1}{7}\sum_{k=8}^{14} X_k, \quad \frac{1}{8}\sum_{k=8}^{15} X_k.$$

Our first new result reads as follows.

Theorem 1. *A sequence* $(X_k : k = 1, 2, \dots)$ *of random variables obeys the SLLN, that is,* (1.2) *is satisfied if and only if*

$$P[\lim_{m\to\infty} \sigma_m^{(1)} = 0] = P[\lim_{m\to\infty} \sigma_m^{(2)} = 0] = 1. \tag{2.3}$$

We note that if only one of the conditions in (2.3) is satisfied, then the sequence (X_k) may fail to obey the SLLN. This is illustrated in the next two examples.

Example 3. Let $X_1 = X_2 := 0$ and

$$X_k := \begin{cases} 2^{2\mu}, & \text{if } k = 2^{2\mu} - 1, \\ -2^{2\mu}, & \text{if } k = 2^{2\mu}, \\ 0, & \text{if } 2^{2\mu} < k < 2^{2(\mu+1)} - 1 \text{ for } \mu = 1, 2, \dots. \end{cases}$$

It is easy to check that

$$\sum_{k=m}^{2m-1} X_k = \begin{cases} 2^{2\mu}, & \text{if } m = 2^{2\mu-1} \text{ for } \mu = 1, 2, \dots; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\sum_{k=m+1}^{2m} X_k = 0 \quad \text{for} \quad m = 1, 2, \dots.$$

Thus, the second condition in (2.3) is satisfied, while the first one is not. Since

$$\sum_{k=1}^{2^{2\mu}-1} X_k = 2^{2\mu} \quad \text{and} \quad \sum_{k=1}^{2^{2\mu}} X_k = 0 \quad \text{for} \quad \mu = 1, 2, \dots,$$

the sequence (X_k) fails to obey the SLLN.

Example 4. Let $X_1 = X_2 := 0$ and

$$X_k := \begin{cases} 2^{2\mu}, & \text{if } k = 2^{2\mu}, \\ -2^{2\mu}, & \text{if } k = 2^{2\mu} + 1, \\ 0, & \text{if } 2^{2\mu} + 1 < k < 2^{2(\mu+1)} \text{ for } \mu = 1, 2, \dots. \end{cases}$$

In this case, we have

$$\sum_{k=m}^{2m-1} X_k = 0 \quad \text{for} \quad m = 1, 2, \dots,$$

and

$$\sum_{k=m+1}^{2m} X_k = \begin{cases} 2^{2\mu}, & \text{if } m = 2^{2\mu-1} \text{ for } \mu = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the first condition in (2.3) is satisfied, while the second one is not; and the sequence (X_k) fails to obey the SLLN.

Next, given a random field $(X_{k\ell} : k, \ell = 1, 2, \ldots)$, we define the following four moving rectangular averages:

$$\sigma_{mn}^{(11)} := \frac{1}{mn} \sum_{k=m}^{2m-1} \sum_{\ell=n}^{2n-1} X_{k\ell} \quad \text{for} \quad m, n = 1, 2, \ldots,$$

$$\sigma_{1n}^{(21)} := \frac{1}{2n} \sum_{k=1}^{2} \sum_{\ell=n}^{2n-1} X_{k\ell}, \quad \sigma_{mn}^{(21)} := \sum_{k=m+1}^{2m} \sum_{\ell=n}^{2n-1} X_{k\ell} \quad \text{for} \quad \begin{matrix} n = 1, 2, \ldots, \\ m = 2, 3, \ldots, \end{matrix}$$

$$\sigma_{m1}^{(12)} := \frac{1}{2m} \sum_{k=m}^{2m-1} \sum_{\ell=1}^{2} X_{k\ell}, \quad \sigma_{mn}^{(12)} := \frac{1}{mn} \sum_{k=m}^{2m-1} \sum_{\ell=n+1}^{2n} X_{k\ell} \text{ for } \begin{matrix} m = 1, 2, \ldots, \\ n = 2, 3, \ldots, \end{matrix}$$

and

$$\sigma_{11}^{(22)} := \frac{1}{2 \cdot 2} \sum_{k=1}^{2} \sum_{\ell=1}^{2} X_{k\ell}, \quad \sigma_{m1}^{(22)} := \frac{1}{2m} \sum_{k=m+1}^{2m} \sum_{\ell=1}^{2} X_{k\ell},$$

$$\sigma_{1n}^{(22)} := \frac{1}{2n} \sum_{k=1}^{2} \sum_{\ell=n+1}^{2n} X_{k\ell}, \quad \sigma_{mn}^{(22)} := \frac{1}{mn} \sum_{k=m+1}^{2m} \sum_{\ell=n+1}^{2n} X_{k\ell} \quad \text{for} \quad m, n = 2, 3, \ldots.$$

Our second new result reads as follows.

THEOREM 2. *A random field $(X_{k\ell} : k, \ell = 1, 2, \ldots)$ obeys the SLLN, that is,* (1.5) *is satisfied if and only if*

$$P[\text{b}-\lim_{m,n\to\infty} \sigma_{mn}^{(ij)} = 0] = 1 \quad \textit{for} \quad i, j = 1, 2. \tag{2.4}$$

We note that if only three of the conditions in (2.4) are satisfied, then the random field $(X_{k\ell})$ may fail to obey the SLLN. This is shown in the next example.

EXAMPLE 5. Set $X_{k\ell} := X_k X_\ell$, where X_k was defined in Example 3. Clearly, we have

$$\sum_{k=m}^{2m-1} \sum_{\ell=n}^{2n-1} X_{k\ell} = \begin{cases} 2^{2\mu} 2^{2\nu}, & \text{if } m = 2^{2\mu-1} \text{ and } n = 2^{2\nu} - 1 \text{ for } \mu, \nu = 1, 2, \ldots, \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\sum_{k=m+1}^{2m} \sum_{\ell=n}^{2n-1} X_{k\ell} = \sum_{k=m}^{2m-1} \sum_{\ell=n+1}^{2n} X_{k\ell} = \sum_{k=m+1}^{2m} \sum_{\ell=n+1}^{2n} X_{k\ell} = 0 \quad \text{for} \quad m, n = 1, 2, \ldots.$$

Clearly, condition (2.4) is not satisfied in the case when $i = j = 1$; but it is satisfied in the other three cases when $\max\{i, j\} = 2$; and $(X_{k\ell})$ fails to obey the SLLN.

Making use of Examples 3 and 4, while combining them appropriately, we may construct such random fields $(X_{k\ell})$ that condition (2.4) is satisfied except for the case when either $i = 2$, $j = 1$ or $i = 1$, $j = 2$ or $i = j = 2$, respectively; and $(X_{k\ell})$ fails to obey the SLLN.

3. Proof of Theorem 1

Necessity. It is trivial. In fact, suppose that (1.2) is satisfied. By (2.1), we have

$$\sigma_M^{(1)} = \Big(2 - \frac{1}{M}\Big)\sigma_{2M-1} - \Big(1 - \frac{1}{M}\Big)\sigma_{M-1} \to 0 \quad \text{as} \quad M \to \infty;$$

and by (2.2), we have

$$\sigma_M^{(2)} = 2\sigma_{2M} - \sigma_M \to 0 \quad \text{as} \quad M \to \infty.$$

These prove the fulfillment of (2.3).

Sufficiency. We will construct an appropriate decomposition of the single sum $\sum_{k=1}^M X_k$ into the sum of special subsums. Given $M \geq 3$, we proceed as follows:

(i) If $m_0 := M$ is an odd number, then we set

$$m_1 := \frac{m_0 + 1}{2}, \quad \sigma_{m_1}^{(1)} := \frac{1}{m_1} \sum_{k=m_1}^{2m_1-1} X_k, \quad \text{and} \quad M_1 := m_1 - 1. \tag{3.1}$$

(ii) If $m_0 := M$ is an even number, then we set

$$m_1 := \frac{m_0}{2}, \quad \sigma_{m_1}^{(2)} := \frac{1}{m_1} \sum_{k=m_1+1}^{2m_1} X_k, \quad \text{and} \quad M_1 := m_1. \tag{3.2}$$

Next, we repeat the above procedure starting with M_1 in place of M. Again, there are two possibilities:

(iii) If M_1 is an odd number, then we set

$$m_2 := \frac{m_1 + 1}{2}, \quad \sigma_{m_2}^{(1)} := \frac{1}{m_2} \sum_{k=m_2}^{2m_2-1} X_k, \quad \text{and} \quad M_2 := m_2 - 1.$$

(iv) If M_1 is an even number, then we set

$$m_2 := \frac{m_1}{2}, \quad \sigma_{m_2}^{(2)} := \frac{1}{m_2} \sum_{k=m_2+1}^{2m_2} X_k, \quad \text{and} \quad M_2 := m_2.$$

We will continue this procedure until it ends with either $M_r = 2$ or $M_r = 1$ for some integer $r = r(M)$. More precisely, we set

$$\sigma_{m_r}^{(2)} := \frac{1}{2}\sum_{k=1}^{2} X_k \quad \text{if} \quad M_r = 2, \qquad \text{and} \qquad \sigma_{m_r}^{(1)} := \frac{1}{1}\sum_{k=1}^{1} X_k \quad \text{if} \quad M_r = 1.$$

To sum up, the whole prcedure yields the following representation:

$$\frac{1}{M}\sum_{k=1}^{M} X_k = \sum_{p=1}^{r} \frac{m_p}{m_0}\sigma_{m_p}^{(*)} \quad (m_0 := M), \tag{3.3}$$

where each moving average $\sigma_{m_p}^{(*)}$ is of from either $\sigma_{m_p}^{(1)}$ or $\sigma_{m_p}^{(2)}$ defined above.

It is easily seen that

$$m_p \geq m_{p+1} + m_{p+2} + \cdots + m_r \quad \text{for} \quad p = 0, 1, \ldots, r-1. \tag{3.4}$$

Furthermore, we have

$$\frac{1}{2} < \frac{m_p}{m_{p-1}} \leq \frac{2}{3}, \quad \text{if} \quad m_{p-1} \quad \text{is odd,}$$

and

$$\frac{m_p}{m_{p-1}} = \frac{1}{2}, \quad \text{if} \quad m_{p-1} \quad \text{is even,}$$

whence it follows that

$$\frac{1}{2} \leq \frac{m_p}{m_{p-1}} \leq \frac{2}{3} \tag{3.5}$$

and

$$\left(\frac{1}{2}\right)^p \leq \frac{m_p}{m_0} \leq \left(\frac{2}{3}\right)^p \quad \text{for} \quad p = 1, 2, \ldots, r. \tag{3.6}$$

Since m_r equals either 1 or 2, putting $p = r$ in (3.6) gives (recall $m_0 := M$)

$$\frac{\log M}{\log 2} - 1 \leq r \leq \frac{\log M}{\log(3/2)}, \quad M \geq 2. \tag{3.7}$$

Now, suppose that (2.3) is satisfied. The proof of the sufficiency part can be completed by applying the classical Toeplitz theorem (see, e.g., [2, p. 36] and also [4, p. 74]) in the case of representation (3.3). However, we do not formulate Toeplitz's theorem in its general form as it is known in Summability Theory. Instead, we will present a straightforward proof of its sufficiency part adjusted to the notations in our concrete situation.

By (2.3), the limits

$$\lim_{m\to\infty} \sigma_m^{(1)}(\omega) = \lim_{m\to\infty} \sigma_m^{(2)}(\omega) = 0 \tag{3.8}$$

exist with probability 1. Let $\omega \in \Omega$ be such that (3.8) is satisfied. In particular, both sequences are bounded by some constant $K = K(\omega) > 0$:

$$|\sigma_m^{(1)}(\omega)| \le K \quad \text{and} \quad |\sigma_m^{(2)}(\omega)| \le K \quad \text{for} \quad m = 1, 2, \ldots. \tag{3.9}$$

Furthermore, given any $\varepsilon > 0$ there exists an integer $\mu(\varepsilon) \ge 2$ such that

$$|\sigma_m^{(i)}(\omega)| < \varepsilon \quad \text{for} \quad i = 1, 2; \quad \text{and} \quad \Big(\frac{2}{3}\Big)^m < \varepsilon \quad \text{whenever} \quad m \ge \mu(\varepsilon). \tag{3.10}$$

By (3.7), we may choose M large enough, say $M > M_0$, so that $r = r(M) > \mu(\varepsilon)$ in representation (3.3). Making use of (3.3)–(3.6), (3.9) and (3.10), we estimate as follows:

$$\begin{aligned} \Big|\frac{1}{M}\sum_{k=1}^{M} X_k(\omega)\Big| &\le \sum_{p=1}^{r} \frac{m_p}{m_0} |\sigma_{m_p}^{(*)}(\omega)| \\ &\le \sum_{p:m_p \ge \mu(\varepsilon)} \frac{m_p}{m_0}\varepsilon + \sum_{p:m_p<\mu(\varepsilon)} \frac{m_p}{m_0} K \le \varepsilon \sum_{p=1}^{r} \frac{m_p}{m_0} + \frac{K}{M}\sum_{p=\rho}^{r} m_p, \end{aligned} \tag{3.11}$$

where $\rho = \rho(\varepsilon)$ is defined by the condition that $m_\rho < \mu(\varepsilon) \le m_{\rho-1}$. Since (see (3.5)) $m_{\rho-1} \le 2m_\rho < 2\mu(\varepsilon)$, it follows from (3.11) that

$$\Big|\frac{1}{M}\sum_{k=1}^{M} X_k(\omega)\Big| \le \varepsilon \cdot 1 + \frac{K}{M} m_{\rho-1} \le \varepsilon\Big(1 + \frac{2K\mu(\varepsilon)}{M}\Big) < 2\varepsilon, \tag{3.12}$$

provided that M is so large that $M > \max\{M_0, 2K\mu(\varepsilon)/\varepsilon\}$. Since $\varepsilon > 0$ is arbitrary in (3.12), this proves (1.2). The proof of Theorem 1 is complete.

4. Proof of Theorem 2

Necessity. Suppose (1.5) is satisfied. By definition, we have

$$\begin{aligned} \sigma_{MN}^{(11)} &:= \frac{1}{MN}\sum_{k=M}^{2M-1}\sum_{\ell=N}^{2N-1} X_{k\ell} \\ &= \frac{1}{MN}\Big(\sum_{k=1}^{2M-1}\sum_{\ell=1}^{2N-1} - \sum_{k=1}^{M-1}\sum_{\ell=1}^{2N-1} - \sum_{k=1}^{2M-1}\sum_{\ell=1}^{N-1} + \sum_{k=1}^{M-1}\sum_{\ell=1}^{N-1}\Big) X_{k\ell} \\ &= \Big(2 - \frac{1}{M}\Big)\Big(2 - \frac{1}{N}\Big)\sigma_{2M-1,2N-1} - \Big(1 - \frac{1}{M}\Big)\Big(2 - \frac{1}{N}\Big)\sigma_{M-1,2N-1} - \\ &\qquad - \Big(2 - \frac{1}{M}\Big)\Big(1 - \frac{1}{N}\Big)\sigma_{2M-1,N-1} + \Big(1 - \frac{1}{M}\Big)\Big(1 - \frac{1}{N}\Big)\sigma_{M-1,N-1} \\ &\to 0 \quad \text{as } M, N \to \infty; \end{aligned}$$

furthermore, we also have that

$$\sup_{M,N\ge 1} |\sigma_{MN}^{(11)}| \le 9 \sup_{M,N\ge 1} |\sigma_{MN}| < \infty.$$

An analogous reasoning works for $\sigma_{MN}^{(21)}$, $\sigma_{MN}^{(12)}$ and $\sigma_{MN}^{(22)}$, too. These prove the fulfillment of (2.4).

SUFFICIENCY. The proof of the sufficiency part hinges on an appropriate decomposition of the double sum $\sum_{k=1}^{\mu}\sum_{\ell=1}^{N} X_{k\ell}$ into the sum of special rectangular subsums, analogously to the decomposition of the single sum $\sum_{k=1}^{M} X_k$ in the proof of Theorem 1 above.

To enter into details, let $M, N \geq 3$ be given. We set $m_0 := M$ and define m_1 by (3.1) and (3.2), according to whether m_0 is odd or even. Analogously, we set $n_0 := N$ and define n_1 in the same manner as in the case of m_1. Depending on the parities of m_0 and n_0, we define one of the following moving rectangular averages:

$$\sigma^{(11)}_{m_1,n_1} := \frac{1}{m_1 n_1} \sum_{k=m_1}^{2m_1-1} \sum_{\ell=n_1}^{2n_1-1} X_{k\ell}, \quad M_1 := m_1 - 1,\ N_1 := n_1 - 1,$$

$$\sigma^{(21)}_{m_1,n_1} := \frac{1}{m_1 n_1} \sum_{k=m_1+1}^{2m_1} \sum_{\ell=n_1}^{2n_1-1} X_{k\ell}, \quad M_1 := m_1,\ N_1 := n_1 - 1,$$

$$\sigma^{(12)}_{m_1,n_1} := \frac{1}{m_1 n_1} \sum_{k=m_1}^{2m_1-1} \sum_{\ell=n_1+1}^{2n_1} X_{k\ell}, \quad M_1 := m_1 - 1,\ N_1 := n_1,$$

$$\sigma^{(22)}_{m_1,n_1} := \sum_{k=m_1+1}^{2m_1} \sum_{\ell=n_1+1}^{2n_1} X_{k\ell}, \qquad M_1 := m_1,\ N_1 := n_1.$$

Next, we repeat the above procedure starting with M_1, N_1 in place of $m_0 := M$ and $n_0 := N$. Then we will repeat it again and again until it ends with either $M_r = 2$ or $M_r = 1$, and either $N_s = 2$ or $N_s = 1$, where $r = r(M)$ and $s = s(N)$; and the final rectangular average will be one of the following four averages:

$$\sigma^{(11)}_{m_r,n_s} := \frac{1}{1}\sum_{k=1}^{1}\sum_{\ell=1}^{1} X_{k\ell}, \quad \sigma^{(21)}_{m_r,n_s} := \frac{1}{2}\sum_{k=1}^{2}\sum_{\ell=1}^{1} X_{k\ell},$$

$$\sigma^{(12)}_{m_r,n_s} := \frac{1}{2}\sum_{k=1}^{1}\sum_{\ell=1}^{2} X_{k\ell}, \quad \sigma^{(22)}_{m_r,n_s} := \frac{1}{4}\sum_{k=1}^{2}\sum_{\ell=1}^{2} X_{k\ell}.$$

To sum up, the whole procedure yields the representation

$$\frac{1}{MN}\sum_{k=1}^{M}\sum_{\ell=1}^{N} X_{k\ell} = \sum_{p=1}^{r}\sum_{q=1}^{s} \frac{m_p n_q}{m_0 n_0} \sigma^{(**)}_{m_p,n_q}, \tag{4.1}$$

where each moving rectangular average $\sigma^{(**)}_{m_p,n_q}$ is of form either $\sigma^{(11)}_{m_p,n_q}$ or $\sigma^{(21)}_{m_p,n_q}$ or $\sigma^{(12)}_{m_p,n_q}$ or $\sigma^{(22)}_{m_p,n_q}$ defined above.

Again, we have (3.4) and its counterpart

$$n_q \geq n_{q+1} + n_{q+2} + \cdots + n_s \quad \text{for} \quad q = 0, 1, \ldots, s-1; \tag{3.4$'$}$$

we conclude (3.5) and its counterpart

$$\frac{1}{2} \le \frac{n_q}{n_{q-1}} \le \frac{2}{3} \quad \text{for} \quad q = 1, 2, \dots, s; \tag{3.5$'$}$$

we have (3.6) and its counterpart

$$\Big(\frac{1}{2}\Big)^q \le \frac{n_q}{n_0} \le \Big(\frac{2}{3}\Big)^q \quad \text{for} \quad q = 1, 2, \dots, s. \tag{3.6$'$}$$

Now, we suppose that (2.4) is satisfied. The proof of the sufficiency part can be completed by applying Robison's theorem (see [3]) in the case of representation (4.1). However, we do not formulate Robison's theorem in its general form as it is known in Summability Theory. Instead, we will present a straightforward proof of its sufficiency part adjusted to the notations in our concrete situation.

By (2.4), each of the four limits

$$\operatorname*{b-lim}_{m,n\to\infty} \sigma^{(ij)}_{mn}(\omega) = 0 \quad \text{for} \quad i, j = 1, 2, \tag{4.2}$$

exists with probability 1. Let $\omega \in \Omega$ be such that (4.2) is satisfied. In particular, each of the four double sequences are bounded by some constant $K = K(\omega) \ge 1$:

$$|\sigma^{(ij)}_{mn}(\omega)| \le K \quad \text{for} \quad m, n = 1, 2, \dots \quad \text{and} \quad i, j = 1, 2. \tag{4.3}$$

Furthermore, given any $\varepsilon > 0$ there exists an integer $\mu(\varepsilon) \ge 2$ so that

$$|\sigma^{(ij)}_{mn}(\omega)| < \varepsilon \quad \text{for} \quad i, j = 1, 2, \quad \text{and} \quad \Big(\frac{2}{3}\Big)^m < \varepsilon \;\; \text{if} \;\; \min\{m, n\} > \mu(\varepsilon). \tag{4.4}$$

Let M and N be large enough, say $\min\{M, N\} > M_0$, so that

$$\min\{r = r(M),\ s = s(N)\} > \mu(\varepsilon)$$

in the representation (4.1). Making use of (3.4)–(3.6), (3.4$'$)–(3.6$'$), (4.3) and (4.4) we estimate as follows:

$$\begin{aligned}
&\Big|\frac{1}{MN}\sum_{k=1}^{M}\sum_{\ell=1}^{N} X_{k\ell}(\omega)\Big| \\
&\quad \le \sum_{p=1}^{r}\sum_{q=1}^{s} \frac{m_p n_q}{m_0 n_0}\Big|\sigma^{(**)}_{m_p,n_q}(\omega)\Big| \\
&\quad \le \sum_{p:m_p\ge\mu(\varepsilon)}\ \sum_{q:n_q\ge\mu(\varepsilon)} \frac{m_p n_q}{m_0 n_0}\varepsilon + \sum_{p:m_p<\mu(\varepsilon)}\ \sum_{q:n_q\ge\mu(\varepsilon)} \frac{m_p n_q}{m_0 n_0}K + \\
&\qquad + \sum_{p:m_p\ge\mu(\varepsilon)}\ \sum_{q:n_q<\mu(\varepsilon)} \frac{m_p n_q}{m_0 n_0}K + \sum_{p:m_p<\mu(\varepsilon)}\ \sum_{q:n_q<\mu(\varepsilon)} \frac{m_p n_q}{m_0 n_0}K \\
&\quad \le \varepsilon \sum_{p=1}^{r}\frac{m_p}{m_0}\sum_{q=1}^{s}\frac{n_q}{n_0} + \frac{K}{M}\sum_{p=\rho_1}^{r} m_p \sum_{q=1}^{s}\frac{n_q}{n_0} + \\
&\qquad + \frac{K}{N}\sum_{p=1}^{r}\frac{m_p}{m_0}\sum_{q=\rho_2}^{s} n_q + \frac{K}{MN}\sum_{p=\rho_1}^{r} m_p \sum_{q=\rho_2}^{s} n_q,
\end{aligned} \tag{4.5}$$

where ρ_1 and ρ_2 are defined by

$$m_{\rho_1} < \mu(\varepsilon) \le m_{\rho_1 - 1} \quad \text{and} \quad n_{\rho_2} < \mu(\varepsilon) \le n_{\rho_2 - 1}.$$

Since (see (3.5) and (3.5′))

$$m_{\rho_1 - 1} \le 2m_{\rho_1} < 2\mu(\varepsilon) \quad \text{and} \quad n_{\rho_2 - 1} \le 2n_{\rho_2} < 2\mu(\varepsilon),$$

it follows from (4.5) that

$$\begin{aligned} \Big| \frac{1}{MN} \sum_{k=1}^{M} \sum_{\ell=1}^{N} X_{k\ell}(\omega) \Big| & \\ & \le \varepsilon \cdot 1 \cdot 1 + \frac{K}{M} m_{\rho_1 - 1} \cdot 1 + \frac{K}{N} \cdot 1 \cdot m_{\rho_2 - 1} + \frac{K}{MN} m_{\rho_1 - 1} n_{\rho_2 - 1} \\ & \le \varepsilon + \frac{2K\mu(\varepsilon)}{M} + \frac{2K\mu(\varepsilon)}{N} + \frac{4K\mu(\varepsilon)^2}{MN} < 4\varepsilon, \end{aligned} \tag{4.6}$$

provided that

$$\min\{M, N\} > \max\Big\{M_0, \frac{2K\mu(\varepsilon)}{\varepsilon}\Big\}.$$

Since $\varepsilon > 0$ is arbitrary in (4.6), this proves (1.5). The proof of Theorem 2 is complete.

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